# 2.10 Cauchy Sequences.

**2.10 A. Definition:** Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then $\{s_n\}_{n=1}^{\infty}$ is called a Cauchy Sequence if for any $\varepsilon > 0$ there exists an $N \in I$ such that

$$|s_m - s_n| < \varepsilon \quad (m, n \geq N)$$

**Theorem 2.10 B:** If the sequence of real numbers $\{s_n\}_{n=1}^{\infty}$ converges, then $\{s_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

(or)

~~To~~ prove that every convergent sequence is a Cauchy sequence.

**Proof:** Given sequence $\{s_n\}_{n=1}^{\infty}$ is a convergent sequence

$\therefore \lim_{n\to\infty} s_n = L$.

$\therefore$ By defn given $\varepsilon > 0$, there exists an $N \in I$ such that $|s_k - L| < \frac{\varepsilon}{2} \quad (k \geq N)$

Thus if $m, n \geq N$

$$|s_m - L| < \varepsilon/2 \quad \text{—— ①}$$

$$|s_n - L| < \varepsilon/2 \quad \text{—— ②}$$

Thus if $m, n \geq N$, we have

$$\begin{aligned}
|s_m - s_n| &= |s_m - L + L - s_n| \\
&\leq |s_m - L| + |L - s_n| \\
&\leq |s_m - L| + |s_n - L| \quad \text{using ① \& ②} \\
&< \varepsilon/2 + \varepsilon/2 \quad \forall\, m, n \geq N.
\end{aligned}$$

$\therefore |s_m - s_n| < \epsilon \quad \forall\, m, n \geq N.$

$\Rightarrow$ sequence $\{s_n\}_{n=1}^{\infty}$ is a Cauchy Sequence.

Hence proved

---

2.10C Lemma: If $\{s_n\}_{n=1}^{\infty}$ is a Cauchy Sequence of real numbers, then $\{s_n\}_{n=1}^{\infty}$ is bounded.

Proof

Given $\{s_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

$\therefore \exists\, N \in I$ such that $|s_m - s_n| < \epsilon \quad \forall\, m, n \geq N$

$$|s_m - s_n| < 1 \quad \forall\, m, n \geq N \qquad \because \epsilon < 1$$

$$\therefore |s_m - s_N| < 1 \quad \forall\, m \geq N$$

Hence if $m \geq N$ we have

$$|s_m| = |(s_m - s_N) + s_N|$$

$$\leq |s_m - s_N| + |s_N|$$

$$|s_m| < 1 + |s_N| \quad (\forall\, m \geq N)$$

If $M = \max\{|s_1|, |s_2|, \ldots |s_{N-1}|\}$

then $|s_m| < M + 1 + |s_N| \quad m \in I$

$$|s_m| < K \qquad \text{where } K = M + 1 + |s_N|$$

$$\forall\, m \in I$$

$\Rightarrow$ sequence $\{s_n\}_{n=1}^{\infty}$ is bounded sequence

---

Theorem 3.10D: If $\{s_n\}_{n=1}^{\infty}$ is a Cauchy Sequence of real numbers, then $\{s_n\}_{n=1}^{\infty}$ is convergent.

Given $\{s_n\}_{n=1}^{\infty}$ is a Cauchy sequence.
given $\epsilon > 0$, there exists $N \in I$ such that

$$|s_m - s_n| < \epsilon/2 \quad (m, n \geq N)$$

$$\Rightarrow |s_N - s_n| < \epsilon/2 \quad \forall\, n \geq N$$

$$|s_n - s_N| < \epsilon/2 \quad \forall\, n \geq N$$

$$-\epsilon/2 < s_n - s_N < \epsilon/2 \quad \forall\, n \geq N$$

$$s_N - \epsilon/2 < s_n < s_N + \epsilon/2 \quad \forall\, n \geq N.$$

$\Rightarrow$ Hence if $n \geq N$ $s_N - \epsilon/2$ is lower bound and $s_N + \epsilon/2$ is upper bound of $\{s_n, s_{n+1}, s_{N+2} \ldots\}$

$\Rightarrow$ This implies for $n \geq N$,

$$s_N - \epsilon/2 \leq g.l.b \{s_n, s_{n+1}, s_{n+2} \ldots\} \leq l.u.b \{s_n, s_{n+1}, \ldots\} \leq s_N + \epsilon/2$$

$$\Rightarrow l.u.b \{s_n, s_{n+1}, s_{n+2} \ldots\} - g.l.b \{s_n, s_{n+1}, s_{n+2} \ldots\} \leq \epsilon$$

$l.u.b\{s_n, s_{n+1}, s_{n+2} \cdots\} \le g.l.b\{s_n, s_{n+1}, s_{n+2} \cdots\} + \epsilon_1$

~~Since~~ $M_n \le m_n + \epsilon_1$

$$\lim_{n\to\infty} M_n \le \lim_{n\to\infty} m_n + \epsilon_1$$

$$\limsup_{n\to\infty} s_n \le \liminf_{n\to\infty} s_n + \epsilon_1$$

since $\epsilon_1$ is arbitrarly small

$$\limsup_{n\to\infty} s_n \le \liminf_{n\to\infty} s_n \quad \text{--- (1)}$$

But always

$$\limsup_{n\to\infty} s_n \ge \liminf_{n\to\infty} s_n \quad \text{--- (2)}$$

using (1) & (2)

$$\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n$$

$$\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = L \text{ (say)}$$

$$\therefore \lim_{n\to\infty} s_n = L$$

$\Rightarrow$ sequence $\{s_n\}_{n=1}^{\infty}$ is convergent sequence.

---

Nested Interval theorem on Sequence.

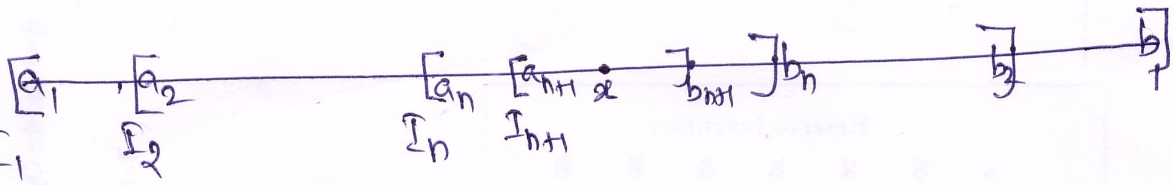
2.10 E Theorem: For each $n \in I$ let $I_n = [a_n, b_n]$ be a bounded interval of real numbers such that $I_1 \supset I_2 \supset \cdots \supset I_n \supset I_{n+1} \supset \cdots$ and $\lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty}(\text{length of } I_n) = 0$. Then $\bigcap_{n=1}^{\infty} I_n$ contains precisely one point.

**Proof**

Given $I_n = [a_n, b_n]$

$\therefore I_1 = [a_1, b_1], \; I_2 = [a_2, b_2] \ldots \; I_n = [a_n, b_n] \ldots$

also given $I_1 \supset I_2 \supset I_3 \ldots I_n \supset I_{n+1} \supset \ldots$




$\Rightarrow a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots$

$\Rightarrow \{a_n\}_{n=1}^{\infty}$ is increasing sequence.

Also $\quad a_n \leq b_1 \quad \forall \; n \in I$

$\Rightarrow \{a_n\}_{n=1}^{\infty}$ is bounded above sequence.

We know that Increasing sequence bounded above is convergent sequence.

$\therefore$ sequence $\{a_n\}_{n=1}^{\infty}$ is convergent sequence.

$$\lim_{n \to \infty} a_n = x \text{ say.}$$

and $\quad b_1 \geq b_2 \geq b_3 \geq \ldots \; b_n \geq b_{n+1} \geq \ldots$

$\Rightarrow$ sequence $\{b_n\}_{n=1}^{\infty}$ is decreasing sequence.

$\forall n \quad b_n \geq a_1$

$\Rightarrow$ sequence $\{b_n\}_{n=1}^{\infty}$ is bounded below

We know that ~~seq of b~~ decreasing sequence and bounded below is convergent

$\therefore$ The sequence $\{b_n\}_{n=1}^{\infty}$ is convergent.

$$\therefore \lim_{n\to\infty} b_n = y \quad (\text{say})$$

But given $\lim_{n\to\infty}(b_n - a_n) = 0$

$$\Rightarrow \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = 0$$

$$y - x = 0$$

$$y = x.$$

$\therefore$ The seq $\{a_n\}_{n=1}^{\infty}$ and seq $\{b_n\}_{n=1}^{\infty}$ converges to the same limit.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x.$$

$$\Rightarrow \quad x \in \bigcap_{n=1}^{\infty} I_n \qquad \because a_n \le x \le b_n \quad \forall n$$

To prove the uniqueness.

Suppose $z \neq x$ such that $z \in \bigcap_{n=1}^{\infty} I_n$

$$x, z \in I_n$$

$$\lim_{n\to\infty} (\text{length of } I_n) = x - z \neq 0$$

which is contradiction

$$\therefore \quad z \in \bigcap_{n=1}^{\infty} I_n$$

$\therefore \bigcap_{n=1}^{\infty} I_n$ contains precisely one point.